J. Fluid Mech. (1987), *vol.* 183, *pp.* 343–363
Printed in Great Britain

On the stability of elliptical vortex solutions of the shallow-water equations

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(Received 9 October 1986 and in revised form 2 March 1987)

The one-layer reduced gravity (or 'shallow water') equations in the f-plane have solutions such that the active layer is horizontally bounded by an ellipse that rotates steadily. In a frame where the height contours are stationary, fluid particles move along similar ellipses with the same revolution period. Both motions (translation along an elliptical path and precession of that orbit) are anticyclonic and their frequencies are not independent; a Rossby number (R_0) based on the combination of both of them is bounded by unity. These solutions may be taken, with some optimism, as a model of ocean warm eddies; their stability is studied here for all values of R_0 and of the ellipse eccentricity (these two parameters determine uniquely the properties of the solution).

Sufficient stability conditions are derived from the integrals of motion; f-plane flows that satisfy them must be either axisymmetric or parallel. For the model vortex, the circular case simply corresponds to a solid-body rotation, and is found to be stable to finite-amplitude perturbations for all values of R_0. This includes $R_0 > \frac{1}{2}$, which implies an anticyclonic absolute vorticity.

The stability of the truly elliptical cases are studied in the normal modes sense. The height perturbation is an n-order polynomial of the horizontal coordinates; the cases for $0 \leqslant n \leqslant 6$ are analysed, for all possible values of the Rossby number and of the eccentricity. All eddies are stable to perturbations with $n \leqslant 2$. (A property of the shallow-water equations, probably related to the last result, is that a general finite-amplitude n-order field is an exact nonlinear solution for $n \leqslant 2$.) Many vortices – noticeably the more eccentric ones – are unstable to perturbations with $n \geqslant 3$; growth rates are $O(R_0^2 f)$ where f is the Coriolis parameter.

1. Introduction

Anticyclonic eddies are a conspicuous feature in the oceans, important for both the physics (see, for instance, the special section of *J. Geophys. Res.* **90**, C5, 1985) and the biology (e.g. Yentsch & Phinney 1985). Their Rossby number, defined as the absolute ratio of the particle swirl speed to the Coriolis parameter, is typically equal to one fourth (Joyce 1984; McWilliams 1985), a moderate value. However, Houghton, Olson & Celone (1986) observed a much more rapidly spinning eddy, with a Rossby number equal to three fourths. This value is so large that the absolute vorticity is anticyclonic; for the simple eddy model used in this work, the Rossby-number upper limit is unity.

Most theoretical models of ocean vortices assume an axisymmetric shape, for the good reason that it is mathematically the easiest to deal with. Indeed, observed eddies are usually quite circular. Joyce *et al.* (1985), for instance, report on a Gulf Stream warm-core ring with a minor to major axis ratio equal to three fourths. Is there a dynamical reason why observed ocean eddies are almost circular? There does not seem to be any *a priori* cause; after all many galaxies are elliptical. (In fact, there could be very eccentric eddies out there, which happen not to have been observed.) Instability could be a good reason for symmetry preference. For instance, a Kirchhoff's vortex is unstable if the aspect ratio is smaller than one third (Love 1893). Griffiths & Pearce (1985) observed a slightly elongated warm eddy with two spiral arms, which was clearly unstable. This work is devoted to the problem of the stability of an elliptical eddy solution of the shallow-water equation (Cushman-Roisin, Heil & Nof 1985), for which, unlike the case of Kirchhoff's vortex, rotation is important.

The model used has, admittedly, a very simple vertical structure: just one active layer. It was chosen because of its mathematical simplicity. The active-layer thickness is proportional to the pressure; since the latter has a maximum for an anticyclonic eddy, the former decreases away from the centre, and the solution is bounded by a zero thickness line. The eddy is, then, limited by a front and therefore the quasi-geostrophic theory cannot be used. Both the basic flow and the perturbation are represented here by polynomials of horizontal position, inside a finite domain; these are quite manageable functions. If the eddy were cyclonic, the thickness could increase away from the centre and the mathematics of the problem would be more cumbersome: this could explain why there is much more literature on anticyclonic than on cyclonic vortices.

The model is formulated in the so called f-plane; the β-term, due to the geoid's curvature, is not considered here. The main effect of β on the eddies is a westward migration at a very slow velocity: of the order of a Rossby-wave phase speed (Nof 1981, 1982; Killworth 1983).

Pressure and velocity fields of the model eddy studied here are seen, in the f-plane, to rotate solidly in the same sense, anticyclonic, but at a slower speed than the fluid particles. Section 2 is devoted to the model equations in a frame where the vortex is seen as steady; the conservation laws are discussed and used to find general stability conditions for finite-amplitude perturbations to any steady nonlinear solution. The elliptical vortex is described in §3, and its stability is studied in the following section. The model eddy is quite simple, and its stability is studied in the whole range of parameter space, i.e. it is not limited to either small Rossby numbers or small eccentricities. This work finds its coordinates in §5: the discussion on the generality of both the model and the results, and on the analogy with other problems is held until this section with the intention of making the presentation clearer. A summary is presented in §6 and the mathematical derivations are confined to two Appendices.

2. Model equations and conservation laws

I start by writing the momentum equations for the so-called shallow-water model in the f-plane:

$$\left.\begin{aligned}\frac{\mathrm{D}u}{\mathrm{D}t}-fv+\frac{\partial p}{\partial x}&=0,\\ \frac{\mathrm{D}v}{\mathrm{D}t}+fu+\frac{\partial p}{\partial y}&=0.\end{aligned}\right\}\tag{1}$$

Here,

$$\frac{\mathrm{D}(\;)}{\mathrm{D}t}\equiv\frac{\partial(\;)}{\partial t}+u\frac{\partial(\;)}{\partial x}+v\frac{\partial(\;)}{\partial y}$$

denotes the material derivative, u and v are velocity components in the x- and y-directions respectively, p is the pressure, and f is the Coriolis parameter, which is taken as constant. The model represents a mass of homogeneous fluid rotating with a vertical angular velocity equal to $\frac{1}{2}f$. However, the centrifugal force does not appear in the f-plane equations because it is balanced by a fraction of the gravitational attraction, effectively tilting the local vertical and changing the local gravity.

Equations (1) are complemented by that for mass conservation:

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0, \qquad (2)$$

where the total depth h is related to p by

$$p=g'h,$$

with g' a 'reduced' gravity. Actually, in this one-layer model h need not be introduced since it can be replaced in (2) by p throughout; the actual value of g' is irrelevant (more on this in §5). Moreover, absolute lengthscales are also unimportant, since (1) and (2) are invariant, in the f-plane, under the transformation $(x,y)\rightarrow\lambda(x,y)$, $(u,v)\rightarrow\lambda(u,v)$, and $p\rightarrow\lambda^2 p$, with arbitrary λ.

Next (and for reasons that will become clear in the following section) I transform variables and equations to a frame with an anticyclonic rotation Ω relative to the f-plane, i.e. with a rotation $\frac{1}{2}f-\Omega$ relative to the fixed stars, where $f\Omega>0$. Variables in this infrarotating frame, which I call the f_* plane, are distinguished by an asterisk subscript, unless they take the same value as they do in the f-plane (like, for instance, the height field). Position and velocity transform as

$$[x_*+\mathrm{i}y_*]=\exp(\mathrm{i}\Omega t)\,[x+\mathrm{i}y] \qquad (3)$$

and

$$[u_*+\mathrm{i}v_*]=\exp(\mathrm{i}\Omega t)\,[(u-\Omega y)+\mathrm{i}(v+\Omega x)]; \qquad (4)$$

this is, of course, the time derivative of (3). The transformed momentum equations are found to be

$$\left.\begin{aligned}\frac{\mathrm{D}u_*}{\mathrm{D}t}-f_*\,v_*+\frac{\partial p}{\partial x_*}&=-\Omega_*^2\,x_*,\\ \frac{\mathrm{D}v_*}{\mathrm{D}t}+f_*\,u_*+\frac{\partial p}{\partial y_*}&=-\Omega_*^2\,y_*,\end{aligned}\right\} \qquad (5)$$

where

$$\frac{\mathrm{D}(\;)}{\mathrm{D}t}\equiv\frac{\partial(\;)}{\partial t_*}+u_*\frac{\partial(\;)}{\partial x_*}+v_*\frac{\partial(\;)}{\partial y_*}.$$

Notice that the Coriolis parameter has been changed so that

$$f_*=f-2\Omega.$$

The centripetal forcing on the right-hand side of (5) comes from the difference of the centrifugal accelerations between the f-plane and the f_* plane; thus

$$\Omega_*^2=(\tfrac{1}{2}f)^2-(\tfrac{1}{2}f_*)^2\equiv\Omega(f-\Omega).$$

Finally, the mass conservation equation is imply

$$\frac{\partial h}{\partial t_*}+\frac{\partial (hu_*)}{\partial x_*}+\frac{\partial (hv_*)}{\partial y_*}=0. \qquad (6)$$

The next step is finding the integrals of motion of the f_* plane equations, (5)–(6), and then using them to derive sufficient stability conditions for any steady flow (in the f_* plane) in the presence of finite-amplitude perturbations. It is important to recall the relationship between those conservation laws and the symmetries of the problem (Ripa 1981; Salmon 1982); this connection will be used to determine the number and type of integrals of motion and to question the usefulness of general stability conditions.

The f_* plane equations, written in the formalism of Hamilton's principle (see Ripa 1981), are invariant under: (i) a general change of particle labels, related to conservation of potential vorticity; (ii) a shift in the origin of time, related to conservation of energy; and (iii) an infinitesimal rotation of the axes (x_*, y_*), related to conservation of angular momentum.

Of course, the f-plane equations are also invariant under translations in both horizontal directions, which results in the conservation of linear momenta; this symmetry is broken in the f_* plane by the presence of the harmonic potential $\frac{1}{2}\Omega_*^2(x_*^2+y_*^2)$. These momenta, equal to the integrals of $h(u-fy)$ and $h(v+fx)$ in the f-plane, have expressions in the f_* plane in which time appears explicitly, through transformations (3) and (4): I am not interested here in this kind of integral of motion, because they do not seem to be useful for obtaining stability conditions with Arnol'd's method.

Ball (1963) studied the integrals of motion for Laplace tidal equations in a rigid basin with the shape of a rotating paraboloid, which – as will be shown in §5 – are mathematically equivalent to the f_* plane equations: both the energy and angular momentum of the centre of mass are also conserved, as well as another quantity C, related to the inertia moment and its rate of change. I have not been able to use these extra constants of motion in the search for stability conditions. In fact, Ball showed that the motion relative to the centre of mass is decoupled from that of the centre itself, and the last constant, C, may not be independent of the others (see Young 1987).

I shall now concentrate on the potential vorticity and the regular energy and angular momentum (Ripa 1981; Salmon 1982). The potential vorticity has the form

$$q = \zeta/h,$$

where

$$\zeta = f_* + \frac{\partial v_*}{\partial x_*} - \frac{\partial u_*}{\partial y_*}$$

$$\equiv f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the absolute vorticity. From the law of conservation of the potential vorticity of each fluid element, $\mathrm{D}q/\mathrm{D}t = 0$, together with that of mass, (6), follows the family of integrals of motion

$$J[u_*, v_*, p] = \int\int \mathrm{d}x\,\mathrm{d}y\, hF(q) = \text{const},$$

where $F(q)$ is an arbitrary function. Total energy and vertical angular momentum are also conserved, viz.

$$E[u_*, v_*, p] = \int\int \mathrm{d}x\,\mathrm{d}y\, h[u_*^2 + v_*^2 + p + \tfrac{1}{2}\Omega_*^2(x_*^2+y_*^2)] = \text{const},$$

and

$$A[u_*, v_*, p] = \int\int \mathrm{d}x\,\mathrm{d}y\, h[x_* v_* - y_* u_* + \tfrac{1}{2}f_*(x_*^2+y_*^2)] = \text{const}.$$

Energy and angular momentum in the f-plane are linear combinations of these integrals in the f_* plane (plus a trivially conserved term) and vice versa.

Let there be some steady solution of the f_* plane equations, viz. $u_* = u_0(x_*, y_*)$, $v_* = v_0(x_*, y_*)$, $p = p_0(x_*, y_*)$: is it possible to say anything *a priori* about its stability? I shall use the conservation laws derived so far in order to obtain sufficient stability conditions, i.e. inequalities that, if satisfied by $[u_0, v_0, p_0]$, writing

$$u_* = u_0(x_*, y_*) + \delta u_*(x_*, y_*, t)$$

and similarly for v_* and p, then some measure of $[\delta u_*, \delta v_*, \delta p]$ is bounded forever. This is done using the pseudoenergy defined by

$$E_p[u_*, v_*, p] = E[u_*, v_*, p] - J[u_*, v_*, p],$$

which is clearly conserved, because so are both E and J, the latter for any form of $F(q)$. This arbitrary function of the potential vorticity is chosen so that the lowest-order contribution (namely, the terms linear in δu_*, etc.) to

$$\delta E_p \equiv E_p[u_0 + \delta u_*, \ldots] - E_p[u_0, \ldots]$$

vanishes identically (the details are given in Appendix A). The stability conditions are those that assure that the next-order contribution, the terms quadratic in the perturbation, is positive definite. Recall that δE_p is a constant of motion; if it is also a sign definite and quadratic functional of $[\delta u_*, \delta v_*, \delta p]$, then the latter cannot grow beyond a limit in any region: the basic flow $[u_0, v_0, p_0]$ is said to be stable in the Lyapunov sense. This method of finding stability conditions is a generalization of that of Arnol'd (1965); its beauty stems from the derivation based on the integrals of motion: it does not involve the 'normal mode equations' (as is the case in §4) and by the same token its results are valid for finite (albeit, small) perturbations, not necessarily infinitesimal ones.

The stability conditions derived from $\delta E_p > 0$ seem to be quite general, since they apply to any steady solution of the f_* plane equations. However, their generality is quite ephemeral: I shall show next that any flow that satisfies them must be axisymmetric. Andrews (1984) first proved a similar result in a different model (quasi-geostrophic flow in the β-plane). For symmetric flows, on the other hand there is another, independent, integral of motion quadratic in the perturbation, which allows for the construction of more powerful stability conditions.

The argument is based on symmetry considerations and proceeds as follows: The f_* plane equations (and, therefore, their constants of motion) are invariant under a fixed rotation in the (x_*, y_*)-plane, $x_* \rightarrow (x_* \cos\alpha - y_* \sin\alpha)$ and $y_* \rightarrow (y_* \cos\alpha + x_* \sin\alpha)$. A particular perturbation may be defined through an infinitesimal rotation of the basic flow, i.e. $u_0(x_*, y_*) + \delta u_*(x_*, y_*) = u_0(x_* - y_* \delta\alpha, y_* + x_* \delta\alpha)$, and similarly for δv_* and δp. A rotated solution must be also a solution; generally a different one but with the same values for the integrals of motion. Thus, (i) $\delta E_p \equiv 0$, because E_p is invariant under rotations, but (ii) if $[u_0, v_0, p_0]$ satisfies the stability conditions, then δE_p should be a positive definite functional of $[\delta u_*, \delta v_*, \delta p]$; therefore (iii) $[\delta u_*, \delta v_*, \delta p]$ must vanish identically, i.e. the flow must be axisymmetric. Namely, for any steady flow that satisfies the stability conditions, (A 3) and (A 4), necessarily

$$u_0 = -\frac{y_*}{r} W(r), \quad v_0 = \frac{x_*}{r} W(r), \quad p_0 = p_0(r),$$

where $r^2 = x_*^2 + y_*^2$. (The momentum balance yields $\mathrm{d}p_0/\mathrm{d}r = f_* W + W^2/r - \Omega_*^2 r$ and the potential vorticity is given by $q_0 = [f_* + \mathrm{d}W/\mathrm{d}r + W/r]/h_0$.)

But if, and only if, the flow is symmetric then a pseudo-angular momentum,

$A_p[u_*, v_*, p]$, can be defined in a similar way to the pseudoenergy: replacing $E[\,]$ by $A[\,]$ and using another $F(q)$. For a non-symmetric flow there is no function $F(q)$ such that the terms in A_p linear in $[\delta u_*, \ldots]$ vanish.

With both E_p and A_p as independent integrals of motion (and quadratic, to lowest order, in the perturbation) a more powerful stability condition can be defined, viz. if there is any value of a parameter σ such that

$$\delta E_p - \sigma \delta A_p > 0$$

for a small perturbation (but with an arbitrary shape), then the basic flow is stable in the Lyapunov sense. The conditions for this to happen are found to be $(u_0 + \sigma y_*)^2 + (v_0 - \sigma x_*)^2 < p_0$ and $(v_0 - \sigma x_*)(\partial q_0/\partial x_*) \geqslant 0$, or equivalently to the last one, $(u_0 + \sigma y_*)(\partial q_0/\partial y_*) \leqslant 0$. Since the flow is axisymmetric, these equations can be rewritten as

$$(W - \sigma r)^2 < p_0 \tag{7}$$

and

$$(W - \sigma r)\frac{\partial q_0}{\partial r} \geqslant 0 \tag{8}$$

for some σ.

Needless to say, a steady axisymmetric solution in the f_* plane is also steady and axisymmetric in the f-plane; conditions (7) and (8) also apply to the latter case. Gordin (1984) found four stability conditions (not a family of couples as here) for the f-plane case, allowing for a radial velocity. I shall argue, at the end of §5, that there is no flow that satisfies Gordin's equations.

It is fundamental to distinguish between stable flows and flows that satisfy the stability condition; the latter constitute a subset of the former. There could be asymmetric stable flows; Andrews' theorem says that their stability cannot be proved using the integrals of motion.

3. The elliptical vortex solution

Here, I shall describe a steady solution of the f_* plane equations that represents an elongated eddy. In the f-plane, then, the vortex has an anticyclonic rotation Ω. The form of the basic fields are $(u_*, v_*, p) = (u_0, v_0, p_0)$ with

$$\left.\begin{aligned} u_0 &= \frac{a}{b}\Omega_* y_*, \\ v_0 &= -\frac{b}{a}\Omega_* x_*, \\ p_0 &= \tfrac{1}{2}ab\Omega_* f_*\left[1 - \left(\frac{x_*}{a}\right)^2 - \left(\frac{y_*}{b}\right)^2\right], \end{aligned}\right\} \tag{9}$$

where a and b are the minor and major semiaxes, respectively, and the sign of Ω_* is equal to that of f_*. These equations are valid inside the ellipse $p_0 = 0$, since p_0 cannot be negative (outside that ellipse there is no fluid of the active layer). This solution was first presented by Cushman-Roisin *et al.* (1985), in f-plane variables, a formula that can be obtained from (9) using the coordinate transformation (3) and (4).

The description of the solution is clearer in the f_* plane than in the f-plane. Each steady solution of the form (9) is fully characterized by two non-dimensional parameters: the ratio of the minor to the major semiaxes with range

$$0 < \frac{a}{b} \leqslant 1, \tag{10}$$

and the ratio of the rotation rate of the eddy to the Coriolis parameter, with values in the range

$$0 < \frac{\Omega}{f} < \tfrac{1}{2}. \tag{11}$$

Any case with $\Omega > \frac{1}{2}f$ is equivalent to that with $\Omega \rightarrow f - \Omega$ and merely results in an opposite sign of f_*. If Ω_*/f_* is used as the second parameter, instead of Ω/f, then the range is between 0 and ∞. The position, orientation, and absolute size of the ellipse are clearly irrelevant degrees of freedom.

There follows next a discussion of the physical properties of this solution. The absolute vorticity is uniform, with value

$$\xi_0 = f_* - \left(\frac{a}{b} + \frac{b}{a}\right)\Omega_*.$$

The divergence of the horizontal velocity field vanishes because the flow is in exactly geostrophic balance,

$$f_* u_* = -\frac{\partial p}{\partial y_*}, \quad f_* v_* = \frac{\partial p}{\partial x_*},$$

in the f_* plane, but not strictly in the f-plane (as shown below). Using this balance in the momentum equations (5) it follows that particle accelerations are produced solely by the harmonic potential $\frac{1}{2}\Omega_*^2(x_*^2 + y_*^2)$, viz.

$$\frac{\mathrm{D}u_*}{\mathrm{D}t} = -\Omega_*^2 x_*, \quad \frac{\mathrm{D}v_*}{\mathrm{D}t} = -\Omega_*^2 y_*.$$

(The f-plane balances are more complicated, viz. $(f-2\Omega)(u-\Omega y) = -\partial p/\partial y$ and $(f-2\Omega)(v+\Omega x) = \partial p/\partial x$, i.e. there is no geostrophic balance (unless $\Omega/f \rightarrow 0$). This, in turn, implies $\mathrm{D}u/\mathrm{D}t = 2\Omega v - \Omega(f-2\Omega)x$ and $\mathrm{D}v/\mathrm{D}t = -2\Omega u - \Omega(f-2\Omega)y$.)

The particle trajectories in the f_* plane are along the ellipses p_0 = const., viz.

$$x_* = a\rho\cos(\Omega_* t + \delta),$$
$$y_* = -b\rho\sin(\Omega_* t + \delta),$$

where ρ, δ are two Lagrangian labels $(0 \leqslant \rho \leqslant 1, 0 \leqslant \delta < 2\pi)$. Thus $2\pi/|\Omega_*|$ is the revolution period of the particles in their elliptical paths, the only motion seen in the f_* plane, whereas $2\pi/|\Omega|$ is the precession period of those orbits, as seen in the f-plane; both rotations are anticyclonic and the latter is the slowest. The Rossby number is defined as

$$R_0 = \frac{\Omega + \Omega_*}{f}.$$

Figure 1 shows examples of particle orbits, both in the f_* plane and the f-plane.

Figure 2 shows the properties of the solution that are only a function of Ω/f, i.e. that are independent of a/b. The deformation radius R is defined by

$$R^2 = \frac{\langle p_0 \rangle}{f^2},$$

with the angle brackets indicating a horizontal average; equivalent expressions are $R^2 = \max(p_0)/2f^2 = ab(\Omega_* f_*/4f^2) = \frac{1}{4}abR_0(1-R_0)$.

The absolute vorticity ξ_0 of any solution in the whole parameter domain, defined by (10) and (11), is shown in figure 3. Notice that it is anticyclonic, $\xi_0 f < 0$, in the unshaded region.

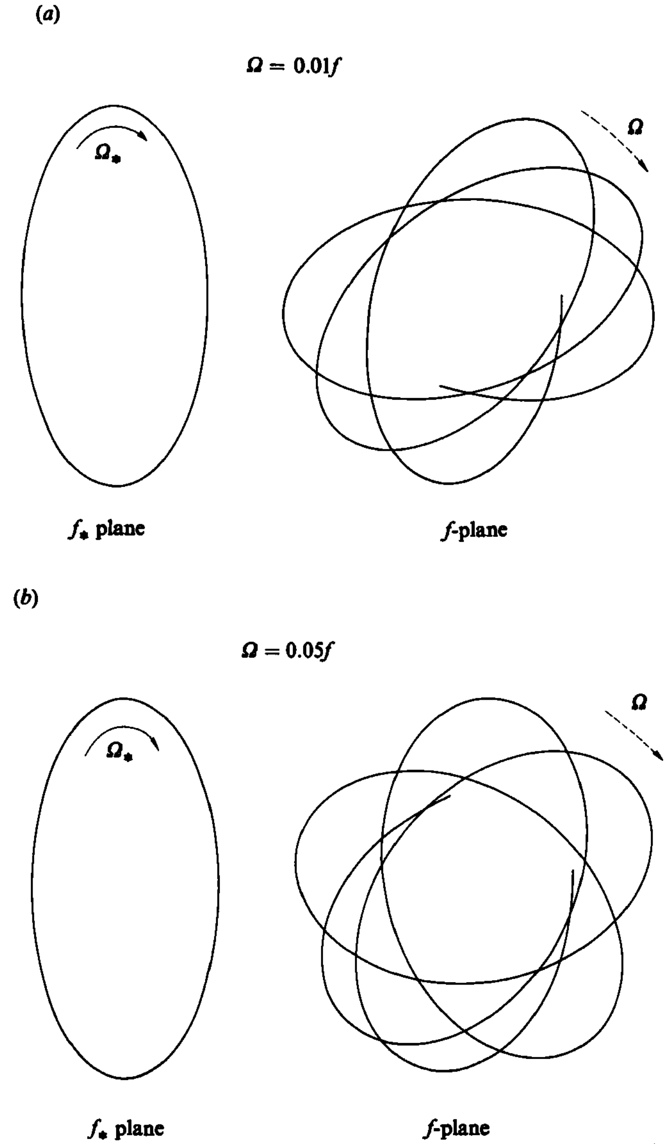

FIGURE 1 (*a*, *b*). For caption see facing page.

4. Stability of the elliptical vortex

I shall now address the problem of the stability of any vortex of the form (9), i.e. for any point in the rectangle of parameter space $(0 < \Omega/f < \frac{1}{2}, 0 < a/b \leqslant 1)$.

The stability conditions derived in §2 from observation laws can only be applied to the circular case, $a = b$, the top boundary of the parameter domain depicted in figure 3. This case is particularly simple: solid-body rotation both in the f_* plane $(u_0 = \Omega_* y_*, v_0 = -\Omega_* x_*)$ and the f-plane $(u = (\Omega+\Omega_*)y, v = -(\Omega+\Omega_*)x)$, and $p_0 = \frac{1}{2}\Omega_* f_*(a^2-r^2)$. It is easily found to be stable to finite-amplitude perturbations using $\sigma = -\Omega_*$ in (7) and (8); Killworth (1983) has already proved its stability to infinitesimal perturbations. Notice that this includes circular vortices with anticyclonic absolute vorticity, $f\xi_0 < 0$, which corresponds to $R_0 > \frac{1}{2}$, i.e. to

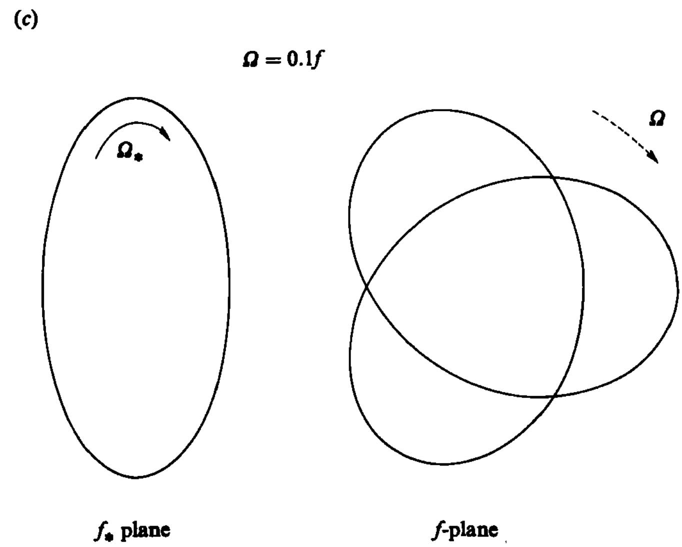

FIGURE 1. Particle trajectory for an elliptical eddy. In the f_* plane (left) the orbit is an ellipse completed in a time $2\pi/|\Omega_*|$. The eddy, and also the orbit, is seen to precede with an angular velocity Ω in the f-plane (right). The aspect ratio a/b equals one-half and the elapsed time equals three f_* plane periods. (*a*) $\Omega/f = 0.01$ which implies $\Omega_*/f = 0.099$ and a Rossby number $R_0 = 0.109$. (*b*) $\Omega/f = 0.05$: $\Omega_* f = 0.22$ and $R_0 = 0.27$. (*c*) $\Omega/f = 0.1$: $\Omega_*/f = 0.3$ and $R_0 = 0.4$.

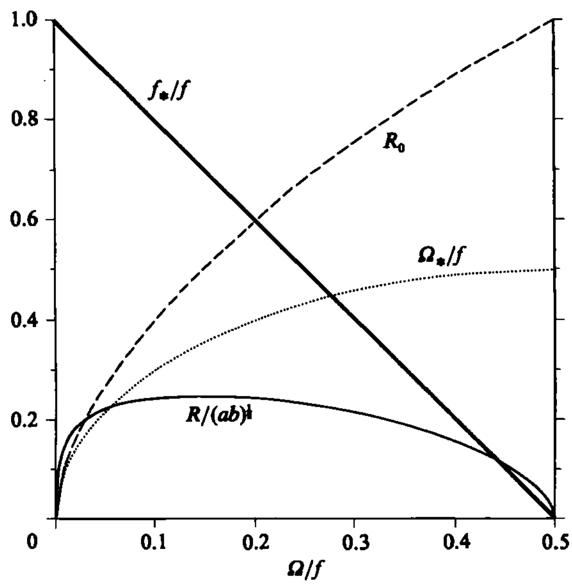

FIGURE 2. Rossby number R_0, Coriolis parameter and revolution frequency in the transformed frame, f_* and Ω_*, and the ratio of the deformation radius to the mean radius of the ellipse, $R/(ab)^{\frac{1}{2}}$. These parameters are only a function of the eddy rotation speed Ω/f, i.e. they are independent of the aspect ratio a/b.

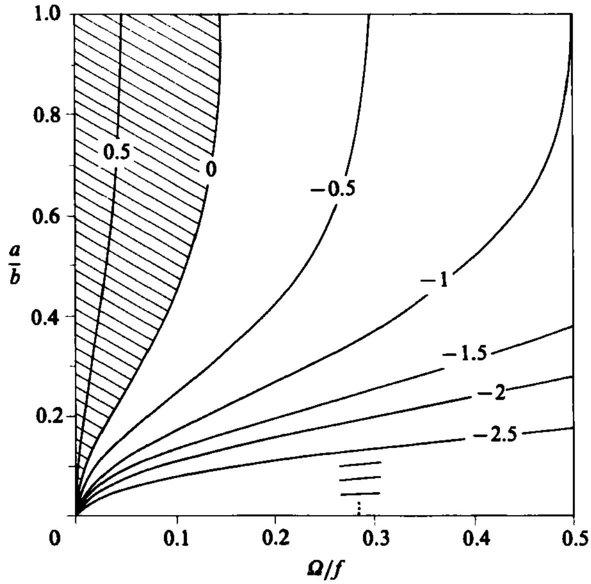

FIGURE 3. Absolute vorticity ξ_0 of the elliptical eddy (normalized by the Coriolis parameter f) as a function of the ratio of the minor and major axes a/b, and the rotation rate Ω of the ellipse.

$\Omega/f > \frac{1}{4}(2-\sqrt{2})$ or $\Omega_*/f_* > \frac{1}{2}$. (Cushman-Roisin *et al.* 1985 incorrectly argued, *a priori*, that these eddies were 'inertially unstable', thereby justifying the choice of the smallest of the two values of R_0 for a given size $(ab)^{\frac{1}{2}}/R$.)

Integrals of motion cannot be used to prove the stability of truly elliptical case ($a \neq b$). Instead, consider the more restricted structure

$$u_* = u_0 + \epsilon \operatorname{Re}[U(x_*, y_*) \exp(-\mathrm{i}\omega t)] + O(\epsilon^2),$$

as $\epsilon \rightarrow 0$, and similarly for v_* and p, with complex amplitudes V and P. Substituting in (5) and (6) and linearizing in ϵ results in the equations

$$\left.\begin{aligned} \mathrm{L}(U) - \left(f_* - \Omega_* \frac{a}{b}\right) V + \frac{\partial P}{\partial x_*} &= 0, \\ \mathrm{L}(V) + \left(f_* - \Omega_* \frac{b}{a}\right) U + \frac{\partial P}{\partial y_*} &= 0, \\ \mathrm{L}(P) + \frac{\partial(p_0 U)}{\partial x_*} + \frac{\partial(p_0 V)}{\partial y_*} &= 0 \end{aligned}\right\} \tag{12}$$

(inside the ellipse $p_0 = 0$), where

$$\mathrm{L}(\;) = -\mathrm{i}\omega + u_0 \frac{\partial(\;)}{\partial x_*} + v_0 \frac{\partial(\;)}{\partial y_*}.$$

The existence of an eigensolution with $\operatorname{Im}(\omega) > 0$ implies instability of the basic flow (u_0, v_0, p_0).

Since the circular case is stable in the Lyapunov sense, it must also be so in the softer normal modes sense: the eigenvalues of (12) must be real for $a = b$. In fact, for this case, the eigensolutions are those of the free oscillations of the surface in a rotating paraboloid (Miles & Ball 1963) Doppler-shifted by the basic flow. The eigenfunctions can be written as either a polynomial of the coordinates x_* and y_*, or as a polynomial of the distance to the centre r times a single harmonic exponential in the azimuthal angle. (The latter is the equivalent of the exp (ikx)-dependence for the well-known problem of parallel-flow stability.) The eigenvalues, real for any value of Ω/f, can be calculated as the roots of

$$\omega'^2+\frac{(n-2s)\,\Omega_* f_* \xi_0}{\omega'} = [n+2s(n-s)]\,\Omega_* f_* + \xi_0^2,$$

where $\omega' = \omega+(n-2s)\,\Omega_*$, and where $n = 0, 1, 2, \ldots,$ and $s = 0, 1, \ldots, n$.

On the other hand, the limit of very eccentric eddies, $a/b \rightarrow 0$ but $\Omega(b/a)^2 \rightarrow$ const., corresponds to a parallel flow with uniform shear, which has been shown to be always unstable by Griffiths, Killworth & Stern (1982). Then, circular eddies are robustly stable and very elliptical ones are unstable; it looks, *a priori*, as if there is a minimum value of b/a for each Ω/f for instability.

For the elliptical case, and given the (x_*, y_*)-dependence of the basic flow, (9), the eigensolution of (12) is written as a polynomial of the coordinates (inside the ellipse $p_0 = 0$). Namely, for a certain integer n,

$$P = \Sigma\, P_{ij}(x_*)^i\,(y_*)^j, \tag{13}$$

where the sum is over all the values (i,j) such that $i+j = n, n-2, \ldots$, etc., all the way down to $i+j = 1$ or 0, depending on the parity of n. U and V have similar structures, but the sum runs over $i+j = n-1, n-3, \ldots$, etc. I call this an n-degrees polynomial solution.

The equations for the highest-order coefficients ($i+j = n$ for P and $i+j = n-1$ for both U and V) are decoupled from those of the lower-order coefficients. Then, the eigenvalues ω are obtained as those of a $(3n+1) \times (3n+1)$ real matrix. The details are given in Appendix B; the main results, particularly those concerning the vortex instability, follow.

The eddy is found to be stable to polynomial perturbations with degree n not greater than two. Moreover, the eigenvalues are exactly given by

$$\omega = 0 \qquad (n = 0),$$

$$\omega = \pm\Omega,\ \pm(f-\Omega) \qquad (n = 1),$$

$$\omega = 0,\ \pm f,\ \pm\left[f^2 \pm 2\Omega_* f_*\left(\frac{a}{b}+\frac{b}{a}\right)\right]^{\frac{1}{2}} \qquad (n = 2).$$

The $n = 0$ solution corresponds to change of the vortex size, and the $n = 1$ ones to a displacement and to inertial oscillations of the centre of mas (Ball 1963).

Some elliptical eddies are unstable to polynomial perturbations with a degree n larger than or equal to three. Figure 4 shows the maximum growth rate, $\mathrm{Im}\,(\omega)$ (normalized by Ω) corresponding to $n = 3$. Notice that $\mathrm{Im}\,(\omega) = O(\Omega)$; in other words,

$$\mathrm{Im}\,(\omega) = O(R_0^2 f).$$

The whole parameter space is presented in this figure; for each value of Ω/f there is a threshold eccentricity b/a for instability. For instance, for $\Omega/f \rightarrow 0$ instability to

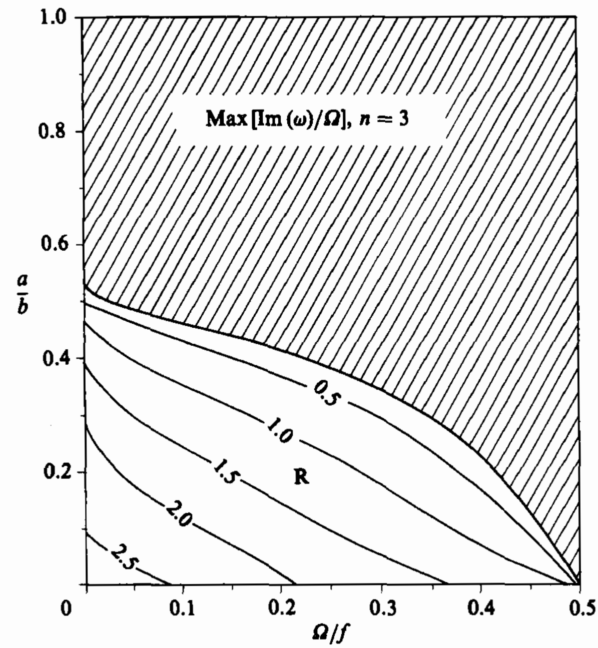

FIGURE 4. Maximum growth rate of an infinitesimal perturbation with $n = 3$, i.e. such that the perturbation pressure is a cubic polynomial of (x_*, y_*), and the perturbation velocities are quadratic polynomials of (x_*, y_*).

cubic perturbations requires $a/b < (\frac{2}{7})^{\frac{1}{2}} \approx 0.53$, whereas for $\Omega = 1/2-\epsilon$ (with $\epsilon \rightarrow 0$) it is necessary that $a/b < \frac{8}{3}\epsilon$. In general, instability requires

$$15\gamma + 9(9\gamma^2+56)^{\frac{1}{2}} < 56C,$$

where

$$\gamma \equiv \frac{\Omega_*}{f_*} = \frac{(\Omega f - \Omega^2)^{\frac{1}{2}}}{f-2\Omega},$$

and

$$C = \tfrac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right).$$

Growing perturbations, $\mathrm{Im}(\omega) > 0$, have $\mathrm{Re}(\omega) = 0$.

The maximum growth rate for perturbations with the next degree, $n = 4$, is shown in figure 5. In the lower part of the figure (the more eccentric eddies), there is an instability region similar to that of the $n = 3$ case; the growth rates are typically twice as large, though. The neutral stability curve for this region, $\mathrm{Im}(\omega) = 0+$ and $\mathrm{Re}(\omega) = 0$, is given by

$$\gamma^2(16C^2-12)+\gamma(17C-9C^3)+1 = 0,$$

which requires an eccentricity $a/b < 0.344100$. In this region, as well as in the lower part of figure 4, $\mathrm{Im}(\omega) > 0$ and $\mathrm{Re}(\omega) = 0$, but not so in the tongue of instability seen to the right of the figure: there, for each value of $\mathrm{Im}(\omega) > 0$, there are two eigensolutions with opposite and non-vanishing values of $\mathrm{Re}(\omega)$.

In figure 4 and subsequent figures regions with a single fastest growing perturbation, with purely imaginary frequency, are denoted by a letter *R*, and regions with two eigensolutions with the same (positive) value of $\mathrm{Im}(\omega)$ but opposite and non-vanishing value of $\mathrm{Re}(\omega)$, are denoted by *G*.

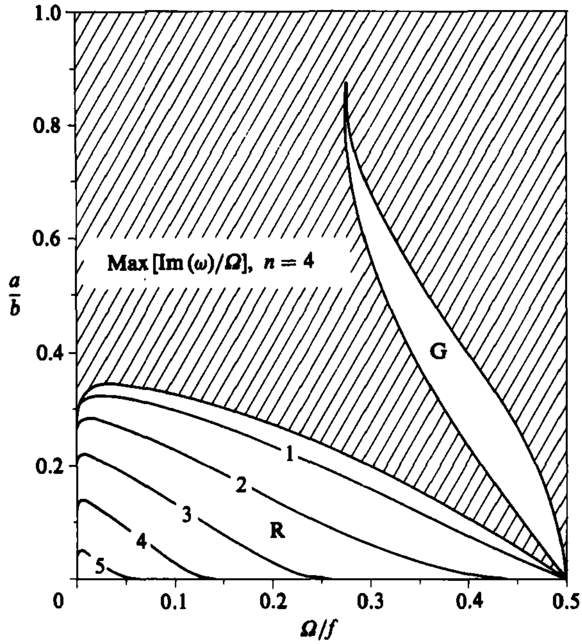

FIGURE 5. As in figure 4 for $n = 4$.

Cushman-Roisin (1986*b*) studied the stability of elliptical eddies of the form (9), in the framework of the 'frontal geostrophic dynamics approximation' (Cushman-Roisin 1986*a*). This approximation is valid in the limit of large and slowly rotating eddies, $R_0 \rightarrow 0$ or $\Omega/f \rightarrow 0$: i.e. the leftmost boundary in figures 4 and 5. Cushman-Roisin's (1986*b*) results coincide with the limit of those of this paper as $\Omega/f \rightarrow 0$, keeping ω/Ω fixed. This seems not to be so in the case of figure 5, because the frontal-dynamics equations predict $\mathrm{Im}\,(\omega/\Omega) = 0$ for even values of n. There is, however, no contradiction: the results of figure 5 are reproduced in figure 6 but using a logarithmic scale of the abscissa in order to expand the low-Ω/f region. Even though $\mathrm{Im}\,(\omega/\Omega)$ strictly vanishes for $\Omega/f = 0$, it has significant non-zero values for very small Ω/f. See, for instance the points along the line $\Omega/f = 0.001$, which corresponds to a very small Rossby number, approximately $R_0 = 0.01$: the growth rates are considerably large, viz. $\mathrm{Im}\,(\omega/\Omega) > 1$ for $b/a > 5$. The neutral stability curve, $\mathrm{Im}\,(\omega) = 0+$, in this limit is given by

$$R_0 \sim \frac{8}{9}\left(\frac{a}{b}\right)^3,$$

i.e. the threshold Rossby number for instability is very small.

These significantly non-zero growth rates for very small values of the Rossby number and even values of the degree n represent an important warning about the practical validity of the frontal-geostrophic-dynamics approximation, which predicts stability.

Maximum growth rates for $n = 5$ and 6 are shown in figures 7 and 8, respectively. The regions of instability for very eccentric eddies are similar to those of figures 4 and 5, which correspond to $n = 3$ and 4, but the growth rates are larger. As n increases, there are more tongues of instability, to both type R and G perturbations,

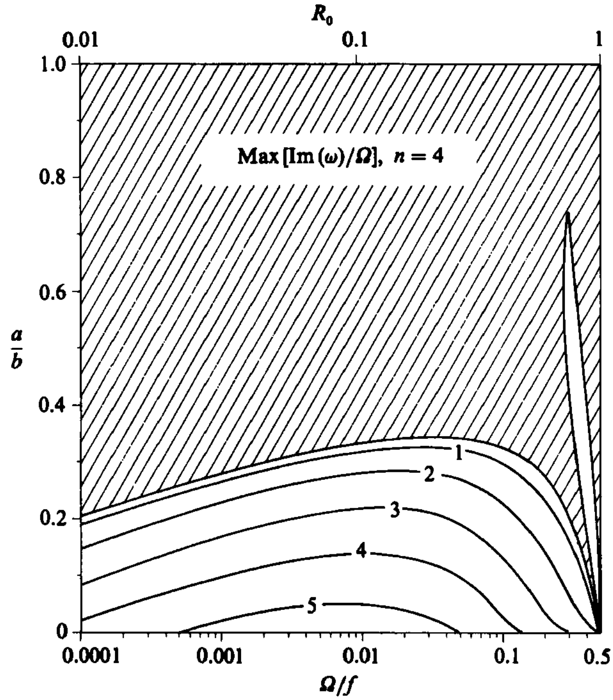

FIGURE 6. As in figure 5, but with logarithmic scale for Ω/f. Even though the growth rate strictly vanishes for $\Omega/f = 0$ (for even n), it takes significant non-zero values for very small values of Ω/f or the Rossby number (the latter parameter is shown as the upper horizontal axis).

that entrain in the zone of less eccentric vortices, the growth rates, Im (ω), are a fraction of Ω.

There seems to be a contradiction between the result that circular vortices are stable to finite-amplitude perturbations, and the tongue of instability in figure 7 ($n = 5$) that touches the $a = b$ line at a point. (I have an analytic expression for the boundary of this particular tongue: there may be others which similarly reach the $a = b$ line but that are not resolved by our numerical method, because they become too sharp.) The apparent paradox is as follows: if there is an elliptical eddy with arbitrarily small eccentricity which is unstable, then the corresponding circular one cannot be stable to finite-amplitude perturbations (since one of them makes it elliptical and unstable). In fact, the slightly eccentric vortex is found (through a linear calculation) to be unstable to infinitesimal perturbations: Arnol'd's theorem assures that, in a truly nonlinear calculation, disturbance cannot grow much, and so the eddy does stay close to its circular shape.

5. Generality of the model

It often happens that identical sets of equations are used to represent different physical models, through a redefinition of the meaning of variables, parameters and constants. This may or may not have some deep significance, but in any case it is important to investigate the generality of the equations at least from a practical point of view to avoid duplicate mathematical work and to make use of results obtained by other authors in other physical contexts. This section is devoted to the discussion of the generality of the model employed in this work and of its results.

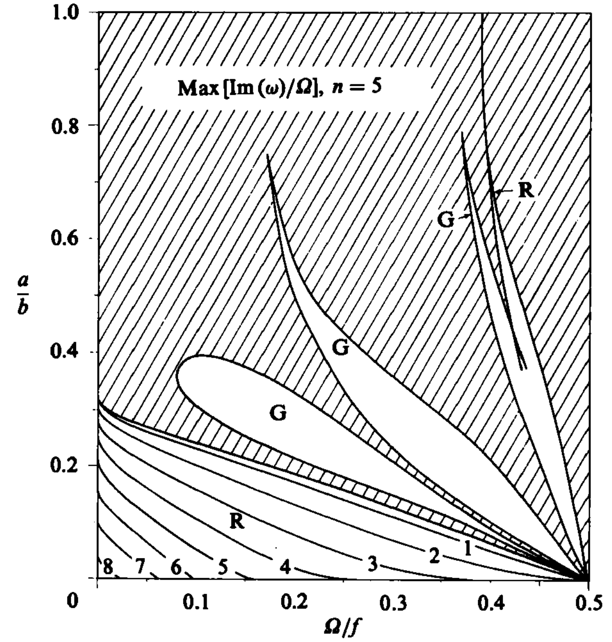

FIGURE 7. As in figure 4 for $n = 5$. The instability regions in the zones of less eccentric eddies are labelled R or G, depending on whether the perturbation has purely imaginary frequency, or there are two eigensolutions with opposite and non-vanishing values of Re (ω), for each Im (ω).

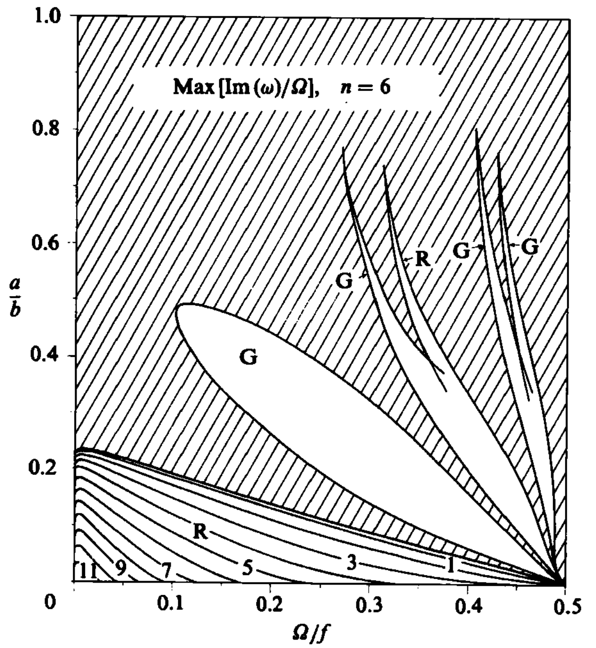

FIGURE 8. As in figure 4 for $n = 6$. The caption of figure 6 applies here to.

I start with the model equations in the f-plane, (1) and (2). These may be taken as Laplace tidal equations, in an ocean with a flat and horizontal, with $p = gh$. Alternatively, the active layer, with thickness h and density ρ, may be sandwiched between upper and lower inert layers, with densities ρ_U and ρ_L; the equations hold with $p = g'h$, where $g' = g(\rho_L - \rho)(\rho - \rho_U)/\rho(\rho_L - \rho_U)$. In particular, it could be that $\rho_U = 0$ (free upper surface) and/or $\rho_L \rightarrow \infty$ (flat level bottom). The point is that since the value of g' is mathematically irrelevant, the same equations can be thought of as representing quite different physical models.

Continuing with the three-layer idea, the upper and lower layers need not be at rest but could have uniform velocities $\boldsymbol{v}_U$ and $\boldsymbol{v}_L$; the equations of motion are the same, with the addition of an advection by an average velocity field $\boldsymbol{v} = [(\rho - \rho_U)\rho_L \boldsymbol{v}_L + (\rho_L - \rho)\rho_U \boldsymbol{v}_U]/(\rho_U - \rho_L)\rho$. The flows in the external layers act as a forcing on the active one through interface tilting. Equivalently, the advection velocity $\boldsymbol{v}$ may be thought of as a manifestation of a sloping flat bottom. In any case, one can easily 'remove' $\boldsymbol{v}$ through a Galilean transformation. This is one example of a simple topography being equivalent to a coordinate transformation.

Consider now the model equations in the f_* plane, (5) and (6), namely, those obtained through a spinning of coordinates. These equations are found to be the same as the f-plane Laplace tidal equations in a basin with the shape of a revolution paraboloid: removing all the asterisk subscripts and writing $p = gh = g(H + \eta)$, where $gH = gH_0 - \frac{1}{2}\Omega_*^2(x^2 + y^2)$, with $gH_0 = \text{const}$. This is another case of a simple topography being equivalent to a transformation of coordinates (see also Ball 1965). Previous research on the revolution paraboloid is related to the present work: I have already shown how the results of Miles & Ball (1963) relate to the circulate-eddy case; Thacker (1981) looked at finite-amplitude perturbations of this system such that the pressure is a quadratic polynomial of the coordinates and the velocities are linear functions of them. In order to be a bit more general, let me define an n-degree field as one such that the pressure is an n-order polynomial of the coordinates and both velocities are $(n-1)$-degree ones (of course, as n-degree field in the f_* plane is an n-degree field in the f-plane and vice versa). The $\frac{1}{2}(n+1)(3n+2)$ coefficients are functions of time: the partial differential equations in the original variables are transformed into ordinary differential equations (ODE) for those coefficients. It is a simple matter of exponent book keeping to convince oneself that an n-degree field is a closed system (i.e. solutions of the system of ODEs are exact nonlinear solutions of the partial ones) if and only if n is not greater than two. (For the β-plane case, no similar result holds, i.e. no finite-degree system is a closed one.) The elliptical eddy in (9) is a steady first-degree solution of the equations for the coefficients; Thacker (1981) found some time-dependent second-degree solutions. (Incidentally, these exact nonlinear solutions constitute a good test of numerical models of the shallow-water equations in a free-boundary domain.) Young (1987) used the integral properties of the system (Ball 1963) to study the global motion of the second-degree field (the most general stable one). There may be some connection, which escapes my knowledge, between the fact that n-degree fields are closed only for n smaller or equal to two and the result of the preceding section that elliptical eddies are unstable (if at all) to n-degree perturbations with n greater than or equal to three.

Finally I should stress the relationship between symmetries and conservation laws. Given a steady solution of the model equations, a pseudoenergy can be constructed to find sufficient conditions for stability to finite-amplitude perturbations. If the basic flow shares a symmetry with the system, though, a pseudomomentum can also be constructed, resulting in more powerful stability conditions (in fact, in a one-parameter family of conditions). Even though the first stability conditions are in principle applicable to very general flows (not necessarily symmetric ones), any flow that satisfies them must be symmetric. This result was first found by Andrews (1984) for the case of quasi-geostrophic three-dimensional flow; its generalization to other geophysical fluid systems (as done here) is trivial, given the connection between conservation laws and symmetries. (Andrews used it to argue that for a non-parallel flow to satisfy the stability conditions in the β-plane there must be some topography that breaks zonal homogeneity.)

The f-plane equations have three symmetries (in addition to time homogeneity): (i) horizontal isotropy, (ii) x-homogeneity, and (iii) y-homogeneity. It is easy to find general sufficient stability conditions for any flow that is steady in the f-plane using pseudoenergy conservation. For a one-layer model, for instance, those are (A 3) and (A 4), where $(u_0, \ldots)$ is measured in the f-plane. Unfortunately, there is no flow that satisfies them because, according to Andrews' theorem, it would have to be (i) axisymmetric, (ii) parallel to the x-axis, and (iii) parallel to the y-axis, at the same time, which is clearly impossible. This is different from saying that there are no stable flows (the circular vortices of §3 are an example of a steady and stable flow in the f-plane). Recall that the key point to obtain this theorem is that x and y do not appear explicitly in the pseudoenergy integral, from which the stability conditions were obtained. Now, suppose we retrict ourselves to flows parallel to the x-axis: then, and only then, there is another integral of motion that can be used to obtain stronger stability conditions, the pseudomomentum along that axis. Andrews' theorem cannot be applied to this integral because 'y' appears explicitly (the momentum per unit mass is equal to $u-fy$ in the f-plane or to $u-f^2/2\beta$ in the β-plane) and thus it is not invariant under either rotations or translations along the y-axis (the stability conditions for the β-plane one-layer model are discussed in Ripa 1983). Therefore it is possible to find symmetric flows that in the f-plane satisfy the stability conditions; the coefficient that multiplies the pseudomomentum must be non-zero. It is not possible to find a flow in the f-plane (which has 'too many' symmetries) that satisfies stability conditions derived from conservation of pseudoenergy alone (such as those of Gordin 1984): pseudomomentum conservation must also be used in order to get stable solutions.

6. Summary

The shallow-water equations transformed to a frame that rotates with anticyclonic angular velocity Ω, relative to the f-plane, show two differences with the original ones: (i) the Coriolis parameter is smaller in magnitude, and (ii) there is a centripetal force of the form $-\Omega_*^2 \boldsymbol{x}$, where $\boldsymbol{x}$ is the position vector and $\Omega_*^2 = \Omega f - \Omega^2$ (f, Ω, and Ω_* all have the same sign). This apparent harmonic forcing, produced by the transformation coordinates, is similar to the real one experienced in a basin (in the f-plane or non-rotating) with the shape of a revolution paraboloid.

The new equations have four integrals of motion independent of the centre-of-mass motion and in which time does not appear explicitly: mass, energy, angular momentum and the volume integral of an arbitrary function of potential vorticity; linear moments are not conserved because the harmonic potential breaks the symmetry of horizontal homogeneity. These conservation laws are used to find sufficient conditions of stability to finite-amplitude perturbations, applicable to any axisymmetric steady solution; the stability of non-symmetric flows cannot be proved by this method.

One particular solution of the transformed equations has all particles traversing similar Lissajous ellipses, i.e. the orbit of a two-dimensional ideal pendulum, with aspect ratio a/b and period $2\pi/|\Omega_*|$. Since elliptical orbits have a precession $-\Omega$ relative to the f-plane, the Rossby number is defined as $R_0 = (\Omega+\Omega_*)/f$, and it is found to be limited by unity. The pressure and (modified) Coriolis forces balance. The global motion is that of an anticyclonic elliptical eddy.

The stability of such eddy is studied in the whole parameter space, $1 \geqslant a/b > 0$ and $1 > R_0 > 0$. The circular case, $a/b = 1$, corresponds to a solid-body rotation and is found to be stable to finite-amplitude perturbations (Lyapunov sense) for all the values of R_0, including the cases of anticyclonic absolute vorticities ($R_0 > \frac{1}{2}$). The stability of the truly elliptical cases, $a/b < 1$, is studied by solving the normal mode equations, which apply to infinitesimal disturbances. The perturbation has the form of a polynomial of the horizontal coordinates with degree equal to n for the pressure and to $(n-1)$ for the velocities. The eddy is stable to perturbations with $n < 3$. Some eddies (notably, the more elliptical ones) are unstable to perturbations with $n \geqslant 3$. The growth rate is of $O(\Omega)$, which yields an e-folding time of a few inertial periods for moderate values of R_0. For $R_0 \rightarrow 0$ the results coincide with those that Cushman-Roisin (1986*b*) found in the less general context of the 'frontal geostrophic dynamics approximation'. For even values of n, however, that theory predicts zero growth rate (which, again, is the result of this work for $R_0 \rightarrow 0$) but significant values of the growth rate are found to Rossby numbers as small as 0.01: the results of the frontal-geostrophic-dynamics approximation should be taken with caution.

I am grateful to Sergio Ramos and Fabián Rosas for drafting the figures and to Sergio Jiménez for assistance with the programming. Miguel Lavin was very helpful in making this a much more readable manuscript. A very stimulating correspondence with Drs Ettore Salusti and Benoit Cushman-Roisin is sincerely appreciated.

Appendix A. General stability conditions

In order to derive the stability conditions, it is convenient to rewrite the momentum equations (5) in the form

$$\left.\begin{aligned}\frac{\partial u_*}{\partial t_*} - \xi v_* + \frac{\partial b}{\partial x_*} = 0,\\ \frac{\partial v_*}{\partial t_*} + \xi u_* + \frac{\partial b}{\partial y_*} = 0,\end{aligned}\right\} \tag{A 1}$$

where ξ is the absolute vorticity and

$$b = p + \tfrac{1}{2}(u_*^2 + v_*^2 + \Omega_*^2 r^2), \tag{A 2}$$

is the Bernoulli head. For a steady solution in the f_* plane, there must exist a transport function Ψ, such that

$$h_0 u_0 = -\frac{\partial \Psi}{\partial y_*}, \quad h_0 v_0 = \frac{\partial \Psi}{\partial x_*}.$$

The above equations further require that both b_0 and q_0 be functions of Ψ, viz. $b_0 = b_0(\Psi)$ and $q_0 = \mathrm{d}b_0/\mathrm{d}\Psi$ (the reader may verify that, in particular, this is true for the elliptical vortex solution described in §3).

Putting now $u = u_0 + \delta u$, etc. the perturbation energy is

$$\delta E = \iint \mathrm{d}x\,\mathrm{d}y[h_0(u_0\,\delta u + v_0\,\delta v) + b_0\,\delta h] + \dots,$$

where $(\dots)$ denotes quadratic plus higher-order terms; clearly δE, which is conserved, is not sign definite. However, consider the family of integrals of motion J, which depend on an arbitrary function of potential vorticity q. Let

$$q = q_0 + \eta + \dots$$

(or exactly $q = q_0 + \eta h_0/h$) where

$$\eta = \left[\frac{\partial(\delta v_*)}{\partial x_*} - \frac{\partial(\delta u_*)}{\partial y_*} - q_0\,\delta h\right]/h_0.$$

For the integrand in the definition of J

$$hF(q) = h_0\,F(q_0) + \delta h[F(q_0) - q_0\,F'(q_0)] + \left[\delta u_*\frac{\partial q_0}{\partial y_*} - \delta v_*\frac{\partial q_0}{\partial x_*}\right]F''(q_0) + \tfrac{1}{2}\eta^2 h_0\,F''(q_0) + \dots.$$

In order to construct a pseudoenergy, $E - J$, without linear terms in $(\delta u_*, \delta v_*, \delta p)$ it is necessary that

$$b_0 = F(q_0) - q_0\,F'(q_0)$$

(which implies $\mathrm{d}\Psi/\mathrm{d}q_0 = -F''(q_0)$). Assume this function is chosen, and consider now the quadratic terms:

$$\delta Ep = \iint \mathrm{d}x\,\mathrm{d}y\left[\left(\delta u_* + \frac{u_0\,\delta h}{h_0}\right)^2 h_0 + \left(\delta v_* + \frac{v_0\,\delta h}{h_0}\right)^2 h_0 + \frac{\delta h^2(p_0 - u_0^2 - v_0^2)}{h_0} + \tfrac{1}{2}\eta^2 h_0\frac{\mathrm{d}\Psi}{\mathrm{d}q_0}\right] + \dots.$$

In order for δEp to be positive for any perturbation,

$$u_0^2 + v_0^2 < p_0, \tag{A 3}$$

and

$$\frac{\mathrm{d}\Psi}{\mathrm{d}q_0} \geqslant 0, \tag{A 4}$$

must hold everywhere.

Now, as explained in the main text, any flow that satisfies these conditions must be axisymmetric, i.e. they are not as general as they look. If, and only if, the flow is axisymmetric a pseudo-angular momentum, $A - J$, can also be constructed in a similar way. Requiring that $\delta Ep - \mathrm{d}\delta Ap$ be positive for any perturbation, conditions (7) and (8) are found, where σ is an arbitrary parameter.

Appendix B

In order to find the eigensolutions of (12) first make

$$U = \mathrm{i}a(A + B),$$

$$V = b(A - B),$$

$$P = abf_*\,C.$$

Then, introduce

$$Z = \frac{x_*}{a} + \mathrm{i}\frac{y_*}{b}$$

and its conjugate Z^* as independent variables, and the two parameters

$$\alpha = \frac{f_*}{\Omega_*}\frac{1}{2}\left(\frac{b}{a}+\frac{a}{b}\right),$$

$$\delta = \frac{1}{2}\left(\frac{f_*}{\Omega_*}\right)\left(\frac{b}{a}-\frac{a}{b}\right),$$

which are equivalent to a/b and Ω/f. Upon substitution in (12) we obtain

$$(\mathrm{L}-\alpha+1)A+\delta B+\alpha\frac{\partial C}{\partial Z^*}+\delta\frac{\partial C}{\partial Z} = 0,$$

$$(\mathrm{L}+\alpha-1)B-\delta A+\alpha\frac{\partial C}{\partial Z}+\delta\frac{\partial C}{\partial Z^*} = 0,$$

$$\mathrm{L}C-\frac{\partial[(1-ZZ^*)A]}{\partial Z}-\frac{\partial[(1-ZZ^*)B]}{\partial Z^*} = 0;$$

where

$$\mathrm{L} = \lambda+Z\frac{\partial(\ )}{\partial Z}-Z^*\frac{\partial(\ )}{\partial Z^*},$$

and

$$\lambda = \omega/\Omega_*$$

is a non-dimensional eigenvalue. Write C as an nth-order polynomial of (x_*, y_*) in the form

$$C = \Sigma\, C_{ms}\, Z^{m-s}\, Z^{*s},$$

where the sum is over $(0 \leqslant m \leqslant n, 0 \leqslant s \leqslant m)$. Similar polynomials are assumed for A and B, but with m going up to $n-1$ instead of n. The equations for the coefficients are

$$(m-s+1)\,A_{s-1}+(\lambda+m-2s)\,C_s+(s+1)\,B_s = (m-s+1)\,A_s+(s+1)\,B_{s+1},$$

$$\alpha(m-s)\,C_s+(\lambda+m-2s-2+\alpha)\,B_s-\delta A_s+\delta(s+1)\,C_{s+1} = 0,$$

$$\delta(m-s)\,C_s+\delta B_s+(\lambda+m-2s-\alpha)\,A_s+\alpha(s+1)\,C_{s+1} = 0.$$

Here the first subscript was omitted for simplicity: it is equal to m for C, to $(m-1)$ for A and B on the left-hand side, and to $(m+1)$ for A and B on the right-hand side. It is understood that $\Psi_{ms} \equiv 0$ if $s < 0$ or $s > m$.

Clearly, only values of m with the same parity as n enter in these equations. For $m = n$, the right-hand side vanishes identically: the equations of the highest-order coefficients are decoupled from the others and therefore the eigenvalues λ are those of a real penta-diagonal $(3n+1)\times(3n+1)$ matrix.

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